Recent Advances in Algorithms Supporting the Polyhedral Model

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- Various libraries such as **ISL, Polylib, Piplib, Omegalib** implement the polyhedral model's underlying mathematical operations.
- The model provides an abstract representation of for-loop nests that enables us to optimize them via different transformations, including: **loop blocking (tiling), loop parallelizing, …**

Polyhedral model (2/2)

The model is based on **statement instances** in a for-loop nest.
It uses four main components to represent the abstraction of a program:

1- **iteration domain**: integer polyhedron of all iteration instances
2- **access relations**: access relations of iteration instances
3- **schedule**: the order of execution
4- **dependency relations**: read/write dependencies.

```
for(i=0; i<2*N+5; i++){
  for(j=0; j<i; j++)
 S: A[i][j] = A[i][j-1]*2;
 /* statement instances:
  {<S,[0,0]>, <S,[0,1]>, ...}
  iteration domain:
  {0 <= i < 2*N+5, 0 <= j < i
   }
  access relations:
  {[i;j], [i;j-1]}
  schedule: lexicographical
  dependency:
  <S,[i,j]> -> <S,[i,j-1]> */

}
```

Overview of the talk

1. Efficient detection of redundancies in systems of linear inequalities (ISSAC 2024) . Joint work with Rui-Juan Jing (Jiangsu University), Yan-Feng Xie (Chinese Academy of Sciences, Beijing) and Chun-Ming Yuan (Chinese Academy of Sciences, Beijing).

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2. Computing the Integer Hull of Convex Polyhedral Sets (CASC 2022) . Joint work with Lin-Xiao Wang (Microsoft).
3. A Pipeline Pattern Detection Technique in Polly (IMPACT 22, LLPP 22) . Joint work with Delaram Talaashrafi (NVIADIA) and Johannes Doerfert (Argonne National Laboratory).

Outline

Efficient detection of redundancies in systems of linear inequalities

Faster computations of integer hulls fo polyhedral sets

A Pipeline Pattern Detection Technique in Polly

$$
\begin{cases}
-x_3 \leq 1 \\
-x_1 - x_2 - x_3 \leq 2 \\
-x_1 + x_2 - x_3 \leq 2 \\
x_1 - x_2 - x_3 \leq 2 \\
x_1 + x_2 - x_3 \leq 2 \\
x_3 0 \leq 1 \\
-x_1 - x_2 + x_3 \leq 2 \\
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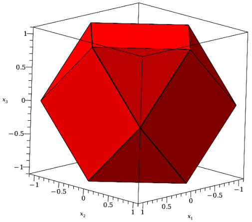

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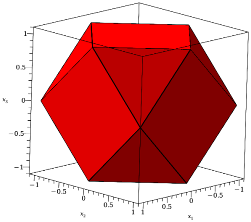

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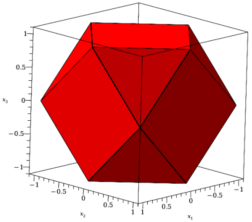

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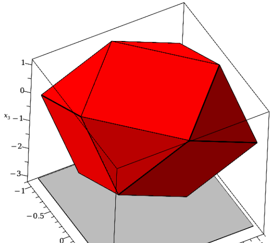

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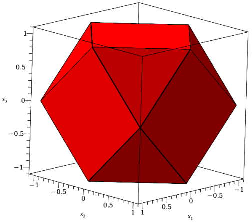

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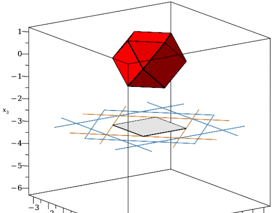

Application of FME: code generation

```
for(i=0; i<=n; i++){
  c[i] = 0; c[i+n] = 0;
  for(j=0; j<=n; j++)
    c[i+j] += a[i]*b[j];
}
```

skip slide

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FME reorders $p > t > i > j > n$ to $i > j > t > p > n$, thus eliminating i, j. skip slide

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The new representation allows us to generate the multithreaded code.

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```
parallel_for (p=0; p<=2*n; p++){
  c[p] = 0;
  for (t=max(0,n-p);
       t<=min(n,2*n-p);t++)
    c[p] += A[t+p-n] * B[n-t];
}
```

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Removing redundant inequalities in FME

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- In our recent paper, we further simplify the redundant detection by manipulating Boolean matrices on which we perform bit-vector arithmetic.

test case	(n, m, k)	mpr	BPAS	cdd	polylib
32hedron	(6, 32, 11)	6.54	16.80	4183.08	**1.92**
64hedron	(7,64,13)	13.05	52.42	>5min	**1.67**
francois	(13,27,2304)	499.92	**253.66**	388.36	> 5min
francois2	(13,31,384)	**41.80**	140.34	55.17	80.63
herve.in	(14,25,262)	34.42	140.34	294.01	**30.08**
c6.in	(11,17,31)	**9.85**	12.72	84.11	5.56
c9.in	(16,18,140)	**25.08**	65.54	151.17	131.53
c10.in	(18,20,142)	22.10	98.68	249.02	**16.06**
S24	(24, 25,25)	23.50	58.80	748.67	**17.47**
S35	(35, 36,36)	46.55	182.14	3575.00	**46.007**
cube	(10, 20,1024)	**81.33**	201.92	125.900	161.06
C56	(5, 6,6)	3.67	4.09	11.81	**0.79**
C1011	(10, 11,11)	24.99	115.68	1716.25	**9.99**
C510	(5, 42,10)	12.00	40.01	>5min	**4.42**
T1	(5, 10,38)	**5.61**	16.44	27.42	8.81
T3	(10,12,29)	21.29	141.64	288.07	**12.07**
T5	(5, 10,36)	8.12	15.62	22.92	**4.76**
T6	(10,20,390)	**1142.9**	23800.11	14937.61	>5min
T7	(5, 8,26)	5.81	10.79	13.96	**4.00**
T9	(10,12,36)	**36.56**	414.53	479.18	100.34
T10	(6, 8,24)	**4.58**	13.65	18.39	5.27
T12	(5, 11,42)	**8.52**	19.03	38.65	8.60
R_15_20	(15, 20,1328)	**28430.40**	336035.00	38037.21	>5min

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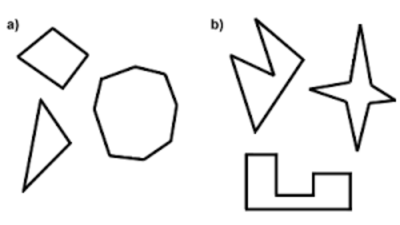
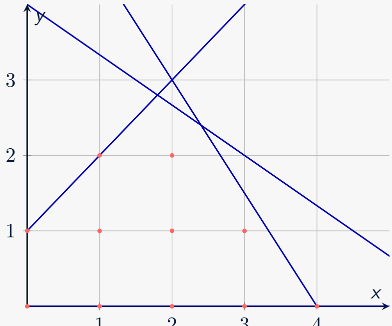
Convex polyhedral sets

- A subset $P \subseteq \mathbb{Q}^n$ is called a *convex polyhedral set* (or simply a *polyhedral set*) if
$$P = \{\mathbf{x} \mid A\mathbf{x} \leq \vec{b}\}$$
holds, for a matrix $A \in \mathbb{Q}^{m \times n}$ and a vector $\vec{b} \in \mathbb{Q}^m$, where n, m are positive integers.
- We are interested in computing P_I the *integer hull* of P that is the smallest convex polyhedral set containing all the integer points of P.

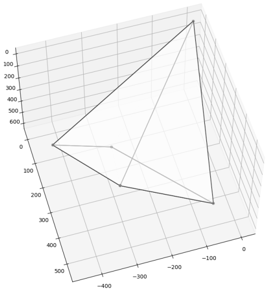




Computing integer hulls (1/3)

The input polyhedral set:

$$\left\{\begin{array}{lcl} -98877x_1 - 189663x_2 - 1798x_3 & \leq & 705915 \\ -10109x_1 - 5958x_2 - 14601x_3 & \leq & 31333 \\ -5405x_1 + 4965x_2 + 3870x_3 & \leq & 4303504 \\ 729x_1 - 117x_2 + 350x_3 & \leq & 4561 \\ 677x_1 + 465x_2 - 540x_3 & \leq & 3489 \end{array}\right.$$



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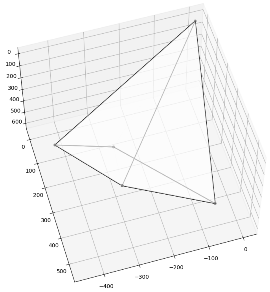
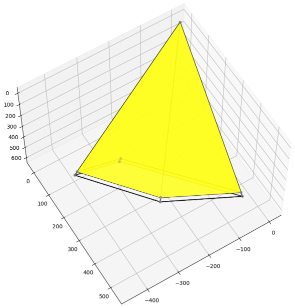
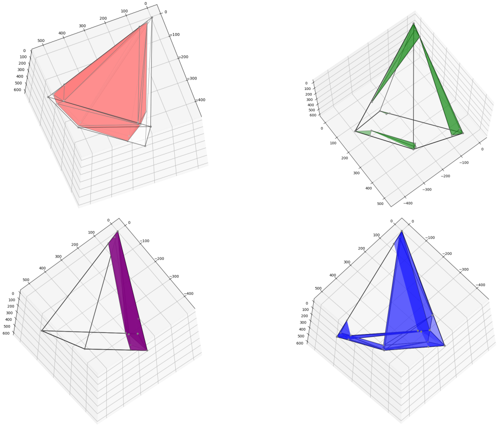


Normalization (leaves the integer hull unchanged):

$$\left\{\begin{array}{rcl} -98877x_1 - 189663x_2 - 1798x_3 & \leq & 705915 \\ -10109x_1 - 5958x_2 - 14601x_3 & \leq & 31333 \\ -1081x_1 + 993x_2 + 774x_3 & \leq & 860700 \\ 729x_1 - 117x_2 + 350x_3 & \leq & 4561 \\ 677x_1 + 465x_2 - 540x_3 & \leq & 3489 \end{array}\right.$$



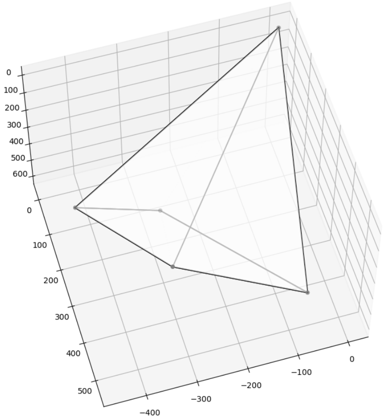
Computing integer hulls (2/3)



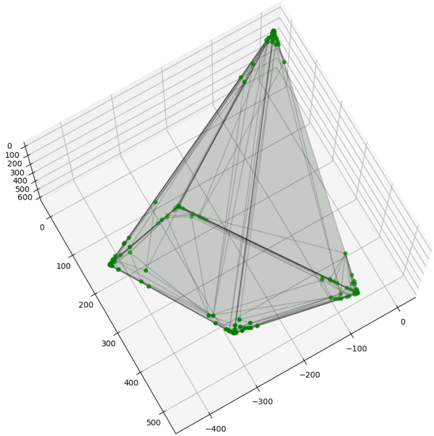
1. The **red** is an approximation of the integer hull of the input.
2. The integer hulls of border regions (green, blue, purple) are brute-force computed via FME.
3. Then `QuickHull` is applied to obtain the integer hull of the input.

Computing integer hulls (3/3)

The input has only 5 vertices.



Its integer hull has 139 vertices.



All details are in https://ir.lib.uwo.ca/etd/8985/ and in https://doi.org/10.1007/978-3-031-14788-3_14

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- This work is about exploiting **cross-loop** parallelization, through tasking.
- It is done by detecting pipeline pattern between **iteration blocks of different loop nests.**
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We use OpenMP, which supports **task parallelization** via:

- `task` construct and `depend` clauses.

Example

```
for(i=0; i<N-1; i++)
  for(j=0; j<N-1; j++)
  S:  A[i][j]=f(A[i][j], A[i][j+1], A[
      i+1][j+1]);

for(i=0; i<N/2-1; i++)
  for(j=0; j<N/2-1; j++)
  R:  B[i][j]=g(A[i][2*j], B[i][j+1],
      B[i+1][j+1], B[i][j]);
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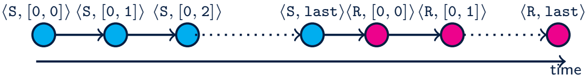

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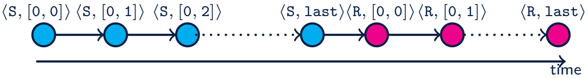

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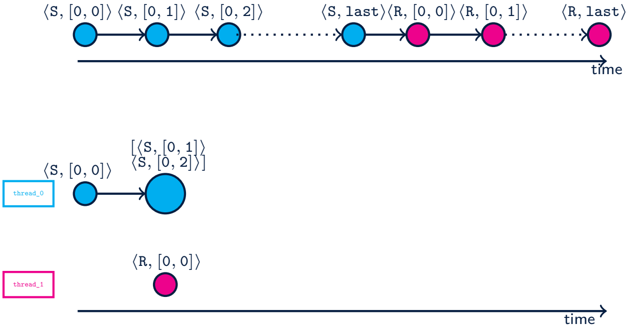

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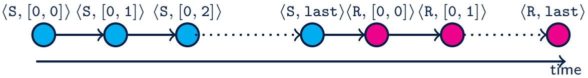

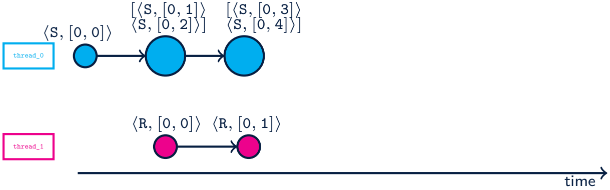

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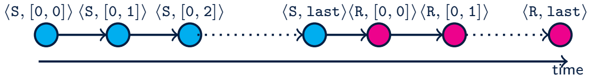

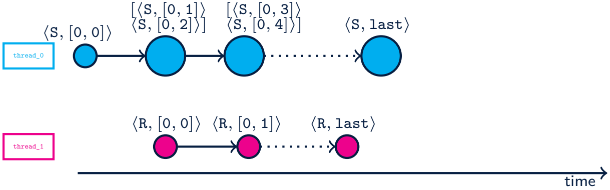

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- The ***composition*** of two maps M_1 and M_2 is denoted by $M_1(M_2)$. It is the set of all pairs $(\vec{i}, \vec{j})$, such that there exists a vector $\vec{k}$, where $(\vec{i}, \vec{k}) \in M_2$ and $(\vec{k}, \vec{j}) \in M_1$.

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- Given two sets S_1 and S_2, the $\mathsf{lexleset}(\mathsf{S}_1, \mathsf{S}_2)$ maps each element $\vec{i} \in S_1$ to all elements $\vec{j} \in S_2$, where $\vec{i}$ is lexicographically less or equal to $\vec{j}$.

Transformation Algorithm (1/5)

Pipeline map

Consider two statements in a program:

- $\mathtt{S}$: iteration domain $\mathcal{I}$, writes in memory location $\mathcal{M}$, $\mathit{Wr}(\mathcal{I} \to \mathcal{M})$
- $\mathtt{T}$: iteration domain $\mathcal{J}$, reads from memory location $\mathcal{M}$, $\mathit{Rd}(\mathcal{J} \to \mathcal{M})$

The **pipeline map** between $\mathtt{S}$ and $\mathtt{T}$ is $\mathcal{T}_{\mathtt{S},\mathtt{T}}(\mathcal{I} \to \mathcal{J})$, where $(\vec{i}, \vec{j}) \in \mathcal{T}_{\mathtt{S},\mathtt{T}}$ if and only if:

1. after running all iterations of $\mathtt{S}$ up to $\vec{i}$, we can safely run all iterations of $\mathtt{T}$ up to $\vec{j}$,
2. $\vec{i}$ is the smallest vector and $\vec{j}$ is the largest vector with Property (1).

Transformation Algorithm (2/5)

Algorithm step I, computing pipeline map and source/target blocking map

1. Relate the iteration domains:
   $$[\mathcal{P}(\mathcal{J} \to \mathcal{I}), \mathcal{P} = Wr^{-1}(Rd)], \mathsf{Domain}(\mathcal{P}) = \mathcal{D}_{\mathcal{P}}$$
2. Map each member of $\mathcal{D}_{\mathcal{P}}$ to all members that are less than or equal to it:
   $$\mathcal{D}'_{\mathcal{P}}(\mathcal{J} \to \mathcal{J})$$
3. Map each $\vec{j} \in \mathcal{J}$ to the largest $\vec{i} \in \mathcal{I}$ that $\vec{j}$ and its previous iterations depend on:
   $$[\mathcal{H}(\mathcal{J} \to \mathcal{I}), \mathcal{H} = \mathsf{lexmax}(\mathcal{P}(\mathcal{D}'))]$$
4. The pipeline map is:
   $$\mathcal{T}_{\mathtt{S},\mathtt{T}} = \mathsf{lexmax}(\mathcal{H}^{-1})$$
5. Partition iteration domain of S (T) with the domain (range) of $\mathcal{T}_{\mathtt{S},\mathtt{T}}$:
   $$\mathcal{B} = \mathsf{Dom}(\mathcal{T}_{\mathtt{S},\mathtt{T}}), \mathcal{B}' = \mathsf{lexleset}(\mathcal{I}, \mathcal{B}), (\mathcal{B} = \mathsf{Range}(\mathcal{T}_{\mathtt{S},\mathtt{T}})\ \mathcal{B}' = \mathsf{lexleset}(\mathcal{J}, \mathcal{B}))$$
6. Compute **source (target) blocking map**:
   $$[\mathcal{V}_{\mathtt{S}}(\mathcal{I} \to \mathcal{I}), \mathsf{lexmin}(\mathcal{B}')], ([\mathcal{Y}_{\mathtt{T}}(\mathcal{J} \to \mathcal{J}), \mathsf{lexmin}(\mathcal{B}')])$$

Transformation Algorithm (3/5)

After finding the pipeline maps between all pairs of dependent statements, we use them to block the iteration domains and construct **pipeline blocking maps**.

The final blocks are such that:

- each block is an atomic task,
- we can establish a pipeline relation between all blocks of all statements,
- maximize the number of blocks of different loops that can execute in parallel.

In the last step, we find **dependency relations** between the tasks.

Transformation Algorithm (4/5)

Algorithm step II, computing pipeline blocking maps

There are several source and target blocking maps associated with each statement.

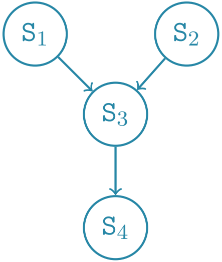
- Minimize the size of the blocks and construct the **optimal blocks**.
- get the lexmin of the union of all source and target blocking maps:
  $\mathcal{E}_{\mathsf{S}} = \mathsf{lexmin}((\bigcup_j(\mathcal{V}^j_{\mathsf{S}}) \cup (\bigcup_i(\mathcal{Y}^i_{\mathsf{S}})))$

Algorithm step III, computing pipeline dependency relations

In a task-parallel program, there are dependency relations between different tasks.

- **Pipeline dependency relations** map each block to the blocks it needs to run correctly.
- For a statement S and a pipeline map $\mathcal{T}_i$, where S is the target:
  $\mathcal{Q}^i_{\mathsf{S}} = \mathcal{T}_i^{-1}(\mathcal{Y}_i(\mathsf{Range}(\mathcal{E}_{\mathsf{S}})))$

Transformation Algorithm (5/5)



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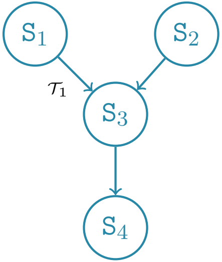
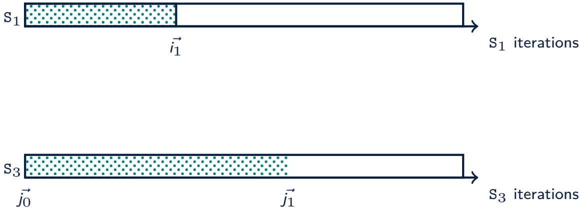
Transformation Algorithm (5/5)





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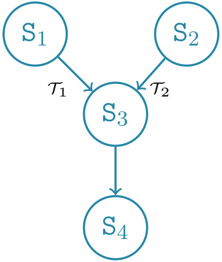

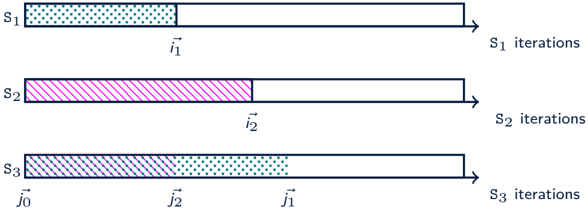

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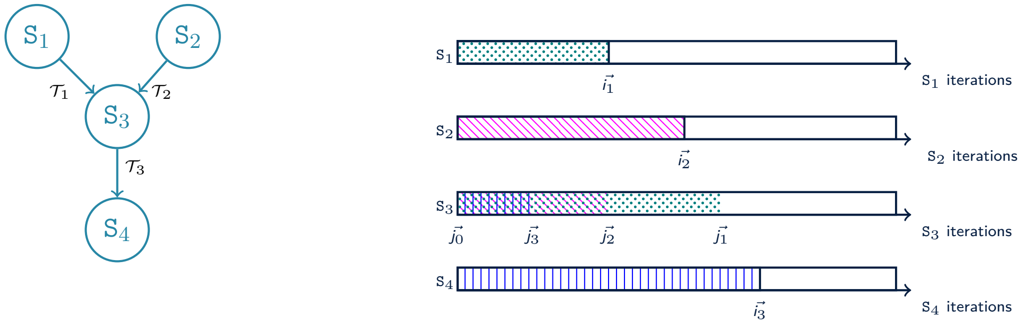

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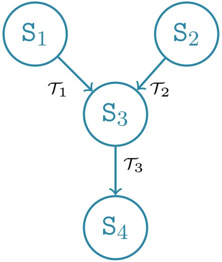

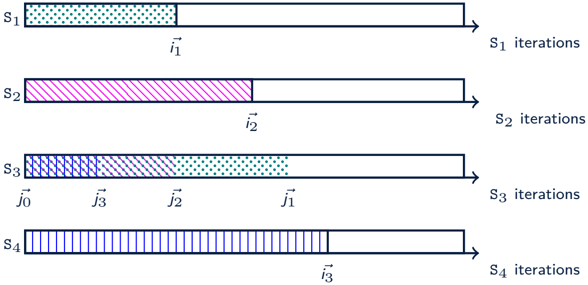

Optimal block of S_3: $\langle \mathrm{S}_3, \vec{j_3} \rangle$
Pipeline dependencies: $\langle \mathrm{S}_1, \vec{i_1} \rangle$, $\langle \mathrm{S}_2, \vec{i_2} \rangle$

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Implementation (1/2)

Analysis passes of Polly

Extend analysis passes of Polly to compute pipeline information for the iteration domains.

Scheduling

1. Create a schedule tree to iterate **over** blocks,
2. Create a schedule tree to iterate **inside** each blocks,
3. **Expand** the first tree with the second tree.
4. Create `pw_multi_aff_list` objects from pipeline dependency relations,
5. Add the `pw_multi_aff_list` objects as mark nodes to the schedule tree.

Implementation (2/2)

Abstract syntax tree

Generate AST from the new schedule tree.
The mark nodes in the schedule tree **annotates** the AST.

Code generation

1. Outline tasks to function calls,
2. Compute unique integer numbers from `pw_multi_aff_list` objects
   - this can be used in OpenMP `depend` clauses.
3. Replace the tasks part in the code with call to the `CreateTask` function that:
   - gets tasks and dependencies, creates OpenMP `tasks` with proper `depend` clauses,
   - handles the order between tasks created from the same loop nest.

Evaluation

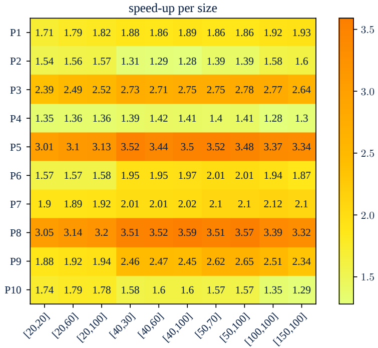

Figure: Speed-up of the tests with different access functions, considering different sizes, comparing sequential version and pipelined version.

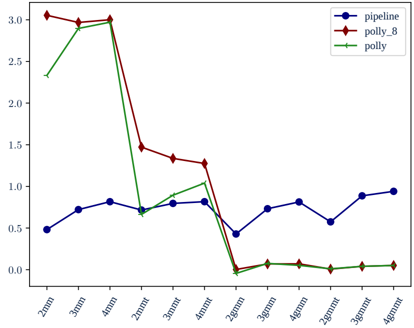

Figure: Comparing logarithm of speed-up gains of Polly running by all available threads, Polly running by `n` threads (`n` is the number of loop nests), and cross-loop pipelining for variants of generalized matrix multiplication.

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Thank You!